

Anderson localization transition with long-ranged hoppings : analysis of the strong multifractality regime in terms of weighted Lévy sums

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For Anderson tight-binding models in dimension d with random on-site energies ϵ_r and critical long-ranged hoppings decaying typically as $V^{typ}(r) \sim V/r^d$, we show that the strong multifractality regime corresponding to small V can be studied via the standard perturbation theory for eigenvectors in quantum mechanics. The Inverse Participation Ratios $Y_q(L)$, which are the order parameters of Anderson transitions, can be written in terms of weighted Lévy sums of broadly distributed variables (as a consequence of the presence of on-site random energies in the denominators of the perturbation theory). We compute at leading order the typical and disorder-averaged multifractal spectra $\tau_{typ}(q)$ and $\tau_{av}(q)$ as a function of q . For $q < 1/2$, we obtain the non-vanishing limiting spectrum $\tau_{typ}(q) = \tau_{av}(q) = d(2q - 1)$ as $V \rightarrow 0^+$. For $q > 1/2$, this method yields the same disorder-averaged spectrum $\tau_{av}(q)$ of order $O(V)$ as obtained previously via the Levitov renormalization method by Mirlin and Evers [Phys. Rev. B 62, 7920 (2000)]. In addition, it allows to compute explicitly the typical spectrum, also of order $O(V)$, but with a different q -dependence $\tau_{typ}(q) \neq \tau_{av}(q)$ for all $q > q_c = 1/2$. As a consequence, we find that the corresponding singularity spectra $f_{typ}(\alpha)$ and $f_{av}(\alpha)$ differ even in the positive region $f > 0$, and vanish at different values $\alpha_+^{typ} > \alpha_+^{av}$, in contrast to the standard picture. We also obtain that the saddle value $\alpha_{typ}(q)$ of the Legendre transform reaches the termination point α_+^{typ} where $f_{typ}(\alpha_+^{typ}) = 0$ only in the limit $q \rightarrow +\infty$.

I. INTRODUCTION

Since its discovery fifty years ago [1], Anderson localization has remained a very active field of research (see the reviews [2–8]). The order parameters of Anderson transitions are the inverse participation ratios (I.P.R.) of eigenfunctions $\psi(\vec{r})$ on a finite volume L^d

$$Y_q(L) \equiv \frac{\int_{L^d} d^d \vec{r} |\psi(\vec{r})|^{2q}}{[\int_{L^d} d^d \vec{r} |\psi(\vec{r})|^2]^q} \quad (1)$$

(the denominator can be omitted if the eigenfunction has been normalized with $\int_{L^d} d^d \vec{r} |\psi(\vec{r})|^2 = 1$). As $L \rightarrow +\infty$, these I.P.R. converge to finite values in the localized phase, and behave as $L^{d(1-q)}$ in the delocalized phase. At criticality, the eigenfunctions become multifractal and the I.P.R. involve non-trivial exponents [6, 8]. It is actually useful to introduce both the typical and the averaged exponents [9]

$$\begin{aligned} Y_q^{typ}(L) &\equiv e^{\overline{\ln Y_q(L)}} && \underset{L \rightarrow \infty}{\simeq} L^{-\tau_{typ}(q)} \\ \overline{Y_q(L)} &&& \underset{L \rightarrow \infty}{\simeq} L^{-\tau_{av}(q)} \end{aligned} \quad (2)$$

The distribution $P_q(y_q)$ of the rescaled variable $y_q = Y_q(L)/Y_q^{typ}(L)$ is expected to decay as a power-law [9, 10]

$$P_q(y_q) \underset{y_q \rightarrow \infty}{\propto} \frac{1}{y_q^{1+\beta_q}} \quad (3)$$

so that the typical and averaged exponents coincide or not according to the value of β_q [8, 9]

$$\begin{aligned} \tau_{typ}(q) &= \tau_{av}(q) && \text{if } \beta_q > 1 \\ \tau_{typ}(q) &\neq \tau_{av}(q) && \text{if } \beta_q < 1 \end{aligned} \quad (4)$$

In particular in the region $q > 0$, one expects that there exists a critical value q_c where $\beta_{q_c} = 1$ separating the region where the two coincide for $q < q_c$ from the region where the two differ with [8, 9]

$$\beta_{q>q_c} = \frac{q_c}{q} \quad (5)$$

For the usual short-ranged Anderson tight-binding model in finite dimension d , one expects a continuous interpolation [8, 11] between a 'weak multifractality' regime obtained in the $d = 2 + \epsilon$ expansion [12] (the leading order

corresponds to the Gaussian parabolic approximation of the multifractal spectrum) and a 'strong multifractality' (SM) that occur in high dimension, with the following limiting form as $d \rightarrow +\infty$

$$\begin{aligned}\tau_{typ}^{SM}(q) &= \tau_{av}^{SM}(q) = d(2q-1) \quad \text{for } q < \frac{1}{2} \\ \tau_{typ}^{SM}(q) &= \tau_{av}^{SM}(q) = 0 \quad \text{for } q > \frac{1}{2}\end{aligned}\tag{6}$$

For Anderson tight-binding models with long-ranged hoppings with typical asymptotic decay

$$V^{typ}(r) \underset{r \rightarrow +\infty}{\propto} \frac{V}{r^a}\tag{7}$$

one has also found a continuous family of the critical points at $a = d$ as a function of the amplitude V [18], that interpolates between 'weak multifractality' for large $V \rightarrow +\infty$ and 'strong multifractality' for small $V \rightarrow 0$ [8]. In dimension $d = 1$, corresponding to the ensemble of $L \times L$ power-law random banded matrices (PRBM), various properties of these Anderson transitions have been studied in detail [9, 19–29]. In particular, the regime $V \rightarrow 0$ has been analyzed via the powerful Levitov renormalization method [9, 30], that allows to compute exactly the disorder-averaged spectrum in the region $q > 1/2$ [9]

$$\tau_{av}^{PRBM}(q) = \frac{4b}{\sqrt{\pi}} \frac{\Gamma(q - \frac{1}{2})}{\Gamma(q - 1)}\tag{8}$$

where $b = V/W$ is the ratio between the amplitude V and the width W of the on-site random energies. The limiting 'strong multifractality' spectrum of Eq. 6 has been then obtained in [9] by using the symmetry of the multifractal spectrum [13–17] connecting exponents with $q < 1/2$ to exponents with $q > 1/2$

$$\tau_{av}(q) - \tau_{av}(1 - q) = d(2q - 1)\tag{9}$$

which is expected to hold for any Anderson transition in the so-called 'conventional symmetry classes' [8]. The derivation of Eq. 6 is thus rather indirect since one performs an explicit calculation in the region $q > 1/2$ to obtain that the exponents vanish $\tau_{av}(q > 1/2) \rightarrow 0$ in this region, and the non-vanishing results in the region $q < 1/2$ are then entirely based on the symmetry of Eq. 9. The same methodology has been followed for related hierarchical models [31] and for the analysis of short-ranged models in high d [11].

The aim of this paper is to reconsider the regime of 'strong multifractality' $V \rightarrow 0$ for Anderson tight-binding models with long-ranged hoppings in order to compute explicitly both the typical and the disorder-averaged multifractal spectra in both regimes $0 < q < 1/2$ and $q > 1/2$. The paper is organized as follows. In section II, we describe the model and apply the standard perturbation theory of quantum mechanics to obtain the leading order expression of the I.P.R. Y_q of Eq. 1. In section III, we analyze the statistical properties of some weighed Lévy sums that play a major role in this perturbation theory. The typical behaviors and the disorder-averaged behaviors of the I.P.R. Y_q are then computed in sections IV and V respectively. Our final results concerning the typical and averaged multifractal spectra in the various regions of q are discussed in section VI and are compared to previous results. Our conclusions are summarized in section VII.

II. PERTURBATION THEORY FOR ANDERSON TIGHT-BINDING MODELS WITH LONG-RANGED HOPPINGS

A. Anderson tight-binding models with long-ranged hoppings

We consider an Anderson tight-binding model on an hypercubic lattice of size L^d defined by the Hamiltonian

$$\begin{aligned}H &= H_0 + H_1 \\ H_0 &= \sum_{\vec{r}} \epsilon_{\vec{r}} |\vec{r}\rangle \langle \vec{r}| \\ H_1 &= \sum_{\vec{r} \neq \vec{r}'} V_{\vec{r}, \vec{r}'} |\vec{r}\rangle \langle \vec{r}'| \end{aligned}\tag{10}$$

where the hoppings are symmetric $V_{\vec{r}', \vec{r}} = V_{\vec{r}, \vec{r}'}$ with $V_{\vec{r}, \vec{r}} = 0$.

1. Assumptions on the random on-site energies $\epsilon_{\vec{r}}$

We consider the usual case where the on-site energies $\epsilon_{\vec{r}}$ are independent random variables distributed with a law $P(\epsilon)$ which is symmetric around $\epsilon = 0$ and which presents the scaling form

$$P(\epsilon) = \frac{1}{W} p\left(\frac{\epsilon}{W}\right) \quad (11)$$

so that W represents the disorder strength. In the following, an important role will be played by the probability density around $\epsilon = 0$ that will be denoted by

$$P(\epsilon = 0) = \frac{c}{2W} \quad (12)$$

where $c = 2p(0) > 0$. For instance in the PRBM model [8], $P(\epsilon)$ is a Gaussian of variance unity corresponding to the values

$$\begin{aligned} W^{PRBM} &= 1 \\ c^{PRBM} &= \sqrt{\frac{2}{\pi}} \end{aligned} \quad (13)$$

The average over the random energies $\{\epsilon_{\vec{r}}\}$ will be denoted by

$$E(A(\{\epsilon_{\vec{r}}\})) \equiv \left[\prod_{\vec{r}} \int d\epsilon_{\vec{r}} P(\epsilon_{\vec{r}}) \right] A(\{\epsilon_{\vec{r}}\}) \quad (14)$$

2. Assumptions on the long-ranged hoppings $V_{\vec{r}, \vec{r}'}$

We consider the critical case where the long-ranged hoppings decay typically as $1/r^d$ in dimension d [1, 18]

$$V(\vec{r}) \underset{r \rightarrow \infty}{\simeq} \frac{V}{r^d} u_{\vec{r}} \quad (15)$$

where $u_{\vec{r}}$ are either fixed ($u_{\vec{r}} = 1$) or are independent random variable of order $O(1)$. For instance in the PRBM model in $d = 1$ [8], $u_{\vec{r}}$ is a Gaussian of variance unity. In any case, we assume here that the distribution of $u_{\vec{r}}$ is such that its moments exist.

In the following, two quantities will play a major role. Denoting by S_d the surface appearing in the radial change of variables $d^d \vec{r} = S_d r^{d-1} dr$ in dimension d , we may evaluate the behavior in L of the following sums

$$\sum_{\vec{r}} \overline{|V(\vec{r})|} = S_d V \overline{|u_{\vec{r}}|} \ln L \quad (16)$$

and for $q < 1/2$

$$\sum_{\vec{r}} \overline{|V(\vec{r})|}^{2q} = V^{2q} \overline{|u_{\vec{r}}|}^{2q} \int S_d r^{d-1} dr \frac{1}{r^{d2q}} = V^{2q} \overline{|u_{\vec{r}}|}^{2q} S_d \frac{L^{d(1-2q)}}{d(1-2q)} \quad (17)$$

B. Perturbation theory in the hoppings

As explained in the Introduction, we focus in this paper on the 'strong multifractality regime' that corresponds to a small amplitude V in the long-ranged hoppings of Eq. 15. It is thus natural to consider the perturbation theory associated to the decomposition of Eq. 10. The Hamiltonian H_0 has for eigenstates the L^d completely localized eigenfunctions on each lattice site, the eigenvalues being simply the corresponding random on-site energies $\epsilon_{\vec{r}}$

$$\begin{aligned} |\phi_{\vec{r}}^{(0)}\rangle &= |\vec{r}\rangle \\ E_{\vec{r}}^{(0)} &= \epsilon_{\vec{r}} \end{aligned} \quad (18)$$

The standard perturbation theory of quantum mechanics yields that at lowest order, the eigenvalues are unchanged

$$E_{\vec{r}}^{(1)} = \epsilon_{\vec{r}} \quad (19)$$

whereas the eigenfunctions read

$$|\phi_{\vec{r}}^{(1)}\rangle = |\phi_{\vec{r}}^{(0)}\rangle + \sum_{\vec{r}' \neq \vec{r}} |\phi_{\vec{r}'}^{(0)}\rangle \frac{\langle \phi_{\vec{r}'}^{(0)} | H_1 | \phi_{\vec{r}}^{(0)} \rangle}{E_{\vec{r}}^{(0)} - E_{\vec{r}'}^{(0)}} = |\vec{r}\rangle + \sum_{\vec{r}' \neq \vec{r}} |\vec{r}'\rangle \frac{V_{\vec{r}, \vec{r}'}}{\epsilon_{\vec{r}} - \epsilon_{\vec{r}'}} \quad (20)$$

The corresponding I.P.R. of Eq. 1 read

$$Y_q^{(1)} = \frac{\sum_{\vec{r}'} |\phi_{\vec{r}'}^{(1)}(\vec{r}')|^{2q}}{\left[\sum_{\vec{r}'} |\phi_{\vec{r}'}^{(1)}(\vec{r}')|^2 \right]^q} = \frac{1 + \Sigma_q}{(1 + \Sigma_1)^q} \quad (21)$$

in terms of the sums

$$\Sigma_q \equiv \sum_{\vec{r}' \neq \vec{r}} \left| \frac{V_{\vec{r}, \vec{r}'}}{\epsilon_{\vec{r}} - \epsilon_{\vec{r}'}} \right|^{2q} \quad (22)$$

To simplify the notations from now on, we will focus on the eigenstate associated to the central site $\vec{r} = 0$ and we will consider that its associated eigenvalue is exactly at the center of the band $\epsilon_0 = 0$. Then the perturbed eigenenergy of Eq. 19 is also $E_0^{(1)} = \epsilon_0 = 0$ at leading order, and the I.P.R. of the corresponding eigenstate (Eq. 21) will then allows to compute at leading order the multifractal spectrum corresponding to $E = 0$. The variables Σ_q of Eq. 22 become

$$\Sigma_q \equiv \sum_{\vec{r} \neq 0} \left| \frac{V(\vec{r})}{\epsilon_{\vec{r}}} \right|^{2q} \quad (23)$$

The aim of this paper is to analyze the statistical properties of the I.P.R. of Eq. 21, in particular their typical values and their disorder-averaged values to extract the multifractal exponents $\tau_{typ}(q)$ and $\tau_{av}(q)$ of Eq. 2. It is convenient to analyze first the statistics of the sums Σ_q that turned out to be weighted Lévy sums as we now explain.

III. STATISTICS OF THE WEIGHTED LÉVY SUMS Σ_q

In this section, we discuss the statistical properties of the sums Σ_q of Eq. 23 that can be rewritten as sums

$$\Sigma_q \equiv \sum_{\vec{r} \in L^d} |V(\vec{r})|^{2q} z_q(\vec{r}) \quad (24)$$

of the random variables

$$z_q(\vec{r}) \equiv |\epsilon_{\vec{r}}|^{-2q} \quad (25)$$

with the weights $|V(\vec{r})|^{2q}$.

We recall here that the average of an observable \mathcal{O} over the random on-site energies will be denoted by $E(\mathcal{O})$ (see Eq. 14). In the case where the long-ranged hoppings of Eq. 15 are non-random ($u_{\vec{r}} \equiv 1$), the disorder-average denoted by $\overline{\mathcal{O}}$ is equal to $E(\mathcal{O})$. In the case where the long-ranged hoppings of Eq. 15 are also random, the disorder-average denoted by $\overline{\mathcal{O}}$ denotes the average over both the random on-site energies and the random variables $u_{\vec{r}}$ appearing in the long-ranged hoppings of Eq. 15.

A. Statistics of the variables $z_q(\vec{r}) \equiv |\epsilon_{\vec{r}}|^{-2q}$

From the probability density $P(\epsilon = 0)$ near zero energy given in Eq. 12, one obtains via a change of variable that the probability distribution $Q_q(z_q)$ of the variable $z_q(\vec{r}) \equiv |\epsilon_{\vec{r}}|^{-2q}$ presents the following power-law decay

$$Q_q(z_q) \underset{z_q \rightarrow +\infty}{\simeq} \frac{c\mu_q}{W z_q^{1+\mu_q}} \quad \text{with} \quad \mu_q = \frac{1}{2q} \quad (26)$$

In particular, the disorder-averaged value $E(z_q)$ (with the notation of Eq. 14) presents a transition at $q = 1/2$

$$\begin{aligned} E(z_q) &< +\infty \quad \text{for } q > \frac{1}{2} \\ E(z_q) &= +\infty \quad \text{for } q \leq \frac{1}{2} \end{aligned} \quad (27)$$

B. Generating function $E(e^{-t\Sigma_q})$ of the sum Σ_q

Lévy sums of identically broadly distributed variables (without the weights $|V(\vec{r})|^{2q}$ in Eq. 24) appears in various fields of disordered systems, usually in low-temperature disorder-dominated phases of classical models: their statistical properties are described in particular in [36–39]. In the following, we analyze the effects of the presence of the weights $|V(\vec{r})|^{2q}$.

The first important property is that the disorder-averaged value $E(\Sigma_q)$ presents the same phase transition as Eq. 27, as a consequence of the linearity of the sum of Eq. 24.

1. Case $q < 1/2$ where the disorder-averaged value $E(\Sigma_q)$ is finite

For $q < 1/2$, the disorder-averaged value is finite and reads

$$E(\Sigma_q) = E(z_q) \sum_{\vec{r} \in L^d} |V(\vec{r})|^{2q} \quad (28)$$

where using Eq. 11

$$E(z_q) = \int_{-\infty}^{+\infty} d\epsilon P(\epsilon) |\epsilon|^{-2q} = W^{-2q} B_q \quad \text{with } B_q = 2 \int_0^{+\infty} dx p(x) x^{-2q} \quad (29)$$

For instance if $p(x)$ is Gaussian of variance unity as in the PRBM model, one obtains

$$B_q^{Gauss} = 2 \int_0^{+\infty} dx \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} x^{-2q} = \frac{\Gamma(\frac{1}{2} - q)}{2^q \sqrt{\pi}} \quad (30)$$

After averaging also over the possibly random hoppings of Eq. 15, we finally obtain using Eq. 17

$$\overline{\Sigma_q} = \left(\frac{V}{W} \right)^{2q} B_q S_d \overline{|u_{\vec{r}}|^{2q}} \frac{L^{d(1-2q)}}{d(1-2q)} \quad (31)$$

2. Case $q > 1/2$ where the disorder-averaged value $E(\Sigma_q)$ is infinite

For $q > 1/2$ where $\mu_q < 1$, the generating function $E(e^{-t\Sigma_q})$ presents the characteristic singularity in t^{μ_q} of Lévy distribution (see Eq. 26)

$$\begin{aligned} E(e^{-t\Sigma_q}) &\equiv \int_0^{+\infty} dz_q Q_q(z_q) e^{-t\Sigma_q} = 1 - \int_0^{+\infty} dz_q Q_q(z_q) (1 - e^{-t\Sigma_q}) \underset{t \rightarrow 0}{\simeq} 1 - t^{\mu_q} \frac{c\mu_q}{W} [-\Gamma(-\mu_q)] + \dots \\ &\underset{t \rightarrow 0}{\simeq} e^{-t^{\mu_q} \frac{c\mu_q}{W} [-\Gamma(-\mu_q)] + \dots} \end{aligned} \quad (32)$$

with the usual integral

$$\int_0^{+\infty} \frac{dx}{x^{1+\mu_q}} (1 - e^{-x}) = -\Gamma(-\mu_q) \quad (33)$$

The generating function $E(e^{-t\Sigma_q})$ will thus presents a similar singularity

$$\begin{aligned} E(e^{-t\Sigma_q}) &= \prod_{\vec{r} \in L^d} E(e^{-t|V(\vec{r})|^{2q} z_q}) \underset{t \rightarrow 0}{\simeq} e^{-\sum_{\vec{r}} (t|V(\vec{r})|^{2q})^{\mu_q} \frac{c\mu_q}{W} [-\Gamma(-\mu_q)]} \\ &\underset{t \rightarrow 0}{\simeq} e^{-t^{\mu_q} \frac{c\mu_q}{W} [-\Gamma(-\mu_q)] \sum_{\vec{r}} |V(\vec{r})|^{2q}} \end{aligned} \quad (34)$$

After averaging also over the possibly random hoppings of Eq. 15, we finally obtain using Eq. 16

$$\begin{aligned} \overline{e^{-t\Sigma_q}} &\underset{t \rightarrow 0}{\simeq} 1 - t^{\mu_q} \frac{c\mu_q}{W} [-\Gamma(-\mu_q)] \sum_{\vec{r}} |\overline{V(\vec{r})}| \\ &\underset{t \rightarrow 0}{\simeq} e^{-t^{\mu_q} \frac{c\mu_q V}{W} [-\Gamma(-\mu_q)] |\overline{u_{\vec{r}}}| S_d \ln L} \end{aligned} \quad (35)$$

Equivalently, inverting Eq. 32, one obtains the following asymptotic decay for the probability distribution $\mathcal{P}_q(\Sigma_q)$

$$\mathcal{P}_q(\Sigma_q) \underset{\Sigma_q \rightarrow +\infty}{\simeq} \frac{\mathcal{A}_q}{\Sigma_q^{1+\mu_q}} \quad (36)$$

with the exponent $\mu_q = 1/(2q)$ and the amplitude

$$\mathcal{A}_q \equiv c S_d \mu_q \frac{V}{W} |\overline{u_{\vec{r}}}| \ln L \quad (37)$$

C. Analysis of the auxiliary quantity $E(z_1^q e^{-tz_1})$

The variables Σ_q and Σ_1 that appear in the numerator and denominator of Eq. 21 are correlated since they involve the same random energies. In the following, some computations will involve the auxiliary quantity

$$E(z_1^q e^{-tz_1}) \quad (38)$$

From Eq. 26, the probability density $Q_1(z_1)$ is known to decay with a power-law of exponent $1 + \mu_1 = 3/2$

$$Q_1(z_1) \underset{z_1 \rightarrow +\infty}{\simeq} \frac{c}{2W z_1^{3/2}} \quad (39)$$

As a consequence for $q < 1/2$, the non-integer moment $E(z_1^q)$ of order q exists and one has

$$E(z_1^q e^{-tz_1}) \underset{t \rightarrow 0}{\simeq} E(z_1^q) + o(t) \quad (40)$$

For $q > 1/2$, the non-integer moment $E(z_1^q)$ of order q does not exist, and the divergence of the auxiliary quantity as $t \rightarrow 0$ can be obtained via the change of variables $x = tz_1$

$$\begin{aligned} E(z_1^q e^{-tz_1}) &\equiv \int_0^{+\infty} dz_1 Q_1(z_1) z_1^q e^{-tz_1} = \int_0^{+\infty} \frac{dx}{t} Q_1\left(\frac{x}{t}\right) \left(\frac{x}{t}\right)^q e^{-x} \\ &\underset{t \rightarrow 0}{\simeq} \frac{c}{2W} t^{\frac{1}{2}-q} \int_0^{+\infty} dx x^{\frac{3}{2}-q} e^{-x} = \frac{c}{2W} t^{\frac{1}{2}-q} \Gamma\left(q - \frac{1}{2}\right) \end{aligned} \quad (41)$$

IV. TYPICAL VALUES OF THE I.P.R. Y_q

To obtain the typical values of the I.P.R. Y_q (see Eq. 2), we need to average the logarithm of the expression of Eq. 21

$$\ln Y_q^{typ} \equiv \overline{\ln Y_q} = \overline{\ln(1 + \Sigma_q)} - q \overline{\ln(1 + \Sigma_1)} \quad (42)$$

So here the correlations between Σ_q and Σ_1 do not play any role, and one only needs to know the statistical properties of the sums Σ_q and Σ_1 discussed in the previous section.

A. Computation of $\overline{\ln(1 + \Sigma_q)}$ for $q < 1/2$

For $q < 1/2$, the disorder-averaged value $\overline{\Sigma_q}$ converges (see Eq. 31). So we may use the expansion of the logarithm to obtain the leading-order behavior

$$\overline{\ln(1 + \Sigma_q)} \simeq \overline{\Sigma_q} = \left(\frac{V}{W}\right)^{2q} B_q S_d |\overline{u_{\vec{r}}}|^{2q} \frac{L^{d(1-2q)}}{d(1-2q)} \quad (43)$$

B. Computation of $\overline{\ln(1 + \Sigma_q)}$ for $q > 1/2$

For $q > 1/2$, we use the following integral representation of the logarithm

$$\ln(1 + \Sigma_q) = \int_0^{+\infty} \frac{dt}{t} e^{-t} (1 - e^{-t\Sigma_q}) \quad (44)$$

to relate the disorder-averaged value to the generating function $\overline{e^{-t\Sigma_q}}$

$$\overline{\ln(1 + \Sigma_q)} = \int_0^{+\infty} \frac{dt}{t} e^{-t} \left(1 - \overline{e^{-t\Sigma_q}}\right) \quad (45)$$

Using the result of Eq. 35, we obtain

$$\begin{aligned} \overline{\ln(1 + \Sigma_q)} &= \int_0^{+\infty} \frac{dt}{t} e^{-t} \left(1 - \overline{e^{-t\Sigma_q}}\right) \simeq \int_0^{+\infty} \frac{dt}{t} e^{-t} \left[t^{\mu_q} \frac{c\mu_q V}{W} [-\Gamma(-\mu_q)] \overline{|u_{\vec{r}}|} S_d \ln L \right] \\ &\simeq S_d \ln L \frac{c\mu_q V}{W} \overline{|u_{\vec{r}}|} [-\Gamma(-\mu_q)] \Gamma(\mu_q) \end{aligned} \quad (46)$$

Using the relation

$$-\Gamma(-\mu_q)\Gamma(\mu_q) = \frac{\pi}{\mu_q \sin(\pi\mu_q)} \quad (47)$$

the final result reads

$$\overline{\ln(1 + \Sigma_q)} \simeq S_d \ln L \frac{cV}{W} \overline{|u_{\vec{r}}|} \frac{\pi}{\sin(\pi\mu_q)} \quad (48)$$

Note that this result can be equivalently obtained via a direct calculation based on the asymptotic behavior of Eq. 36

$$\begin{aligned} \overline{\ln(1 + \Sigma_q)} &= \int_0^{+\infty} d\Sigma_q \mathcal{P}_q(\Sigma_q) \ln(1 + \Sigma_q) \simeq \mathcal{A}_q \int_0^{+\infty} \frac{d\Sigma_q}{\Sigma_q^{1+\mu_q}} \ln(1 + \Sigma_q) = \mathcal{A}_q \frac{\pi}{\mu_q \sin(\pi\mu_q)} \\ &= S_d \ln L \overline{|u_{\vec{r}}|} \frac{cV}{W} \frac{\pi}{\sin(\pi\mu_q)} \end{aligned} \quad (49)$$

C. Behavior of the typical I.P.R. Y_q for $0 < q < 1/2$

For $q < 1/2$, we have seen that $\overline{\ln(1 + \Sigma_q)}$ grows as $L^{d(1-2q)}$ (cf 43), whereas $\overline{\ln(1 + \Sigma_1)}$ grows only as $\ln L$ (Eq 48). We thus obtain at leading order

$$Y_q^{typ}(L) \equiv e^{\overline{\ln Y_q}} \sim e^{\overline{\Sigma_q}} \sim e^{\left(\frac{V}{W}\right)^{2q} S_d B_q \overline{|u_{\vec{r}}|}^{2q} \frac{L^{d(1-2q)}}{d(1-2q)}} \simeq 1 + \left(\frac{V}{W}\right)^{2q} S_d B_q \overline{|u_{\vec{r}}|}^{2q} \frac{L^{d(1-2q)}}{d(1-2q)} \quad (50)$$

The typical exponents $\tau_{typ}(q)$ defined in Eq. 2 thus read

$$\tau_{typ}(q < 1/2) = d(2q - 1) \quad (51)$$

in agreement with the 'strong multifractality' limit of Eq. 6.

D. Behavior of the typical I.P.R. Y_q for $q > 1/2$

For $q > 1/2$, we have seen that both $\overline{\ln(1 + \Sigma_q)}$ and $\overline{\ln(1 + \Sigma_1)}$ grow as $\ln L$ (Eq. 48). These two contributions yield at leading order

$$\overline{\ln Y_q} = \overline{\ln(1 + \Sigma_q)} - q \overline{\ln(1 + \Sigma_1)} = S_d \ln L \frac{cV}{W} \overline{|u_{\vec{r}}|} \left[\frac{\pi}{\sin(\pi\mu_q)} - q \frac{\pi}{\sin(\pi\mu_1)} \right] \quad (52)$$

Since $\mu_1 = 1/2$, we obtain that the typical exponents $\tau_{typ}(q)$ defined in Eq. 2 read

$$\tau_{typ}(q > 1/2) = S_d \frac{cV}{W} \overline{|u_{\vec{r}}|} \pi \left[q - \frac{1}{\sin(\frac{\pi}{2q})} \right] \quad (53)$$

V. DISORDER-AVERAGED VALUES OF THE I.P.R. Y_q

To compute the disorder-averaged values of the I.P.R. Y_q of Eq. 21, it is convenient to use the identity

$$\frac{1}{a^q} = \frac{1}{\Gamma(q)} \int_0^{+\infty} dt t^{q-1} e^{-at} \quad (54)$$

to obtain

$$\begin{aligned} \overline{Y_q} &= \overline{Y_q}|_{\text{first contribution}} + \overline{Y_q}|_{\text{second contribution}} \\ \overline{Y_q}|_{\text{first contribution}} &= \frac{1}{\Gamma(q)} \int_0^{+\infty} dt t^{q-1} e^{-t} \overline{e^{-t\Sigma_1}} \\ \overline{Y_q}|_{\text{second contribution}} &= \frac{1}{\Gamma(q)} \int_0^{+\infty} dt t^{q-1} e^{-t} \overline{\Sigma_q e^{-t\Sigma_1}} \end{aligned} \quad (55)$$

We now evaluate separately these two contributions.

A. Computation of the first contribution

Using Eq. 35 for $q = 1$ with $\mu_1 = 1/2$ and $[-\Gamma(-1/2)] = 2\sqrt{\pi}$

$$\overline{e^{-t\Sigma_1}} \simeq e^{-t^{1/2} \frac{cV\sqrt{\pi}}{W} \overline{|u_{\vec{r}}|} S_d \ln L} \quad (56)$$

the first contribution of Eq. 55 reads at leading order

$$\begin{aligned} \overline{Y_q}|_{\text{first contribution}} &\equiv \frac{1}{\Gamma(q)} \int_0^{+\infty} dt t^{q-1} e^{-t} \overline{e^{-t\Sigma_1}} \simeq \frac{1}{\Gamma(q)} \int_0^{+\infty} dt t^{q-1} e^{-t} \left(1 - t^{1/2} \frac{cV\sqrt{\pi}}{W} \overline{|u_{\vec{r}}|} S_d \ln L \right) \\ &\simeq 1 - \frac{\Gamma(q + \frac{1}{2})}{\Gamma(q)} \frac{cV\sqrt{\pi}}{W} \overline{|u_{\vec{r}}|} S_d \ln L \end{aligned} \quad (57)$$

B. Computation of the second contribution

To evaluate the second contribution of Eq. 55, we first need to evaluate, using the definitions of Eqs 24 and 25

$$\begin{aligned} E(\Sigma_q e^{-t\Sigma_1}) &= E\left(\sum_{\vec{r}} |V(\vec{r})|^{2q} z_1^q(\vec{r}) e^{-t\Sigma_{\vec{r}}} |V(\vec{r}')|^{2q} z_1^q(\vec{r}')\right) \\ &= \sum_{\vec{r}} |V(\vec{r})|^{2q} E\left(z_1^q(\vec{r}) e^{-t|V(\vec{r})|^2 z_1^q(\vec{r})}\right) E\left(e^{-t\Sigma_{\vec{r}' \neq \vec{r}}} |V(\vec{r}')|^{2q} z_1^q(\vec{r}')\right) \end{aligned} \quad (58)$$

For $q < 1/2$, we use Eqs 40 and 56 to obtain

$$E(\Sigma_q e^{-t\Sigma_1}) \underset{t \rightarrow 0}{\simeq} E(z_1^q) \sum_{\vec{r}} |V(\vec{r})|^{2q} \quad (59)$$

so that the second contribution reads at leading order using Eq. 17

$$\overline{Y_q}|_{\text{second contribution}} \simeq \overline{\Sigma_q} = E(z_1^q) \sum_{\vec{r}} \overline{|V(\vec{r})|^{2q}} \simeq E(z_1^q) S_d V^{2q} \overline{|u_{\vec{r}}|^{2q}} \frac{L^{d(1-2q)}}{d(1-2q)} \quad (60)$$

For $q > 1/2$, we use Eqs 41 and 56 to evaluate the singularity of Eq. 58 for small t

$$\begin{aligned} E(\Sigma_q e^{-t\Sigma_1}) &\underset{t \rightarrow 0}{\simeq} \sum_{\vec{r}} |V(\vec{r})|^{2q} \frac{c}{2W} (tV^2(\vec{r}))^{\frac{1}{2}-q} \Gamma\left(q - \frac{1}{2}\right) \left(1 - t^{1/2} \frac{cV\sqrt{\pi}}{W} \overline{|u_{\vec{r}}|} S_d \ln L + \dots\right) \\ &\underset{t \rightarrow 0}{\simeq} t^{\frac{1}{2}-q} \sum_{\vec{r}} |V(\vec{r})|^{2q} \frac{c}{2W} \Gamma\left(q - \frac{1}{2}\right) + \dots \end{aligned} \quad (61)$$

Using Eq. 16, we finally obtain at leading order

$$\begin{aligned}\overline{Y_{q>1/2}}|_{\text{secondcontribution}} &= \frac{1}{\Gamma(q)} \int_0^{+\infty} dt t^{q-1} e^{-t\overline{\Sigma_q} e^{-t\overline{\Sigma_1}}} \simeq \frac{1}{\Gamma(q)} \frac{c}{2W} \Gamma(1/2) \Gamma\left(q - \frac{1}{2}\right) \sum_{\vec{r}} \overline{|V(\vec{r})|} \\ &\simeq \frac{\Gamma\left(q - \frac{1}{2}\right)}{\Gamma(q)} \frac{c\sqrt{\pi}}{2W} S_d V \overline{|u_{\vec{r}}|} \ln L\end{aligned}\quad (62)$$

C. Disorder-averaged I.P.R. for $q < 1/2$

For $q < 1/2$, the first contribution of order $\ln L$ (Eq. 57) is negligible with respect to the second contribution of Eq. 60 which leads to

$$\overline{Y_{q<1/2}} \simeq E(z_1^q) S_d V^{2q} \overline{|u_{\vec{r}}|^{2q}} \frac{L^{d(1-2q)}}{d(1-2q)} \quad (63)$$

The disorder-averaged exponents $\tau_{av}(q)$ defined in Eq. 2 thus read

$$\tau_{av}(q < 1/2) = d(2q - 1) = \tau_{typ}(q < 1/2) \quad (64)$$

and coincide the the typical exponents $\tau_{typ}(q)$, in agreement with the 'strong multifractality' limit of Eq. 6.

D. Disorder-averaged I.P.R. for $q > 1/2$

For $q > 1/2$, we add the two contributions of order $\ln L$ obtained in Eqs 57 and 62. Using $\Gamma(z+1) = z\Gamma(z)$, this leads to

$$\begin{aligned}\overline{Y_{q>1/2}} &= 1 + \left[\frac{\Gamma\left(q - \frac{1}{2}\right)}{\Gamma(q)} - 2 \frac{\Gamma\left(q + \frac{1}{2}\right)}{\Gamma(q)} \right] \frac{c\sqrt{\pi} S_d V}{2W} \overline{|u_{\vec{r}}|} \ln L \\ &= 1 - \frac{\Gamma\left(q - \frac{1}{2}\right)}{\Gamma(q-1)} \frac{c\sqrt{\pi} S_d V}{W} \overline{|u_{\vec{r}}|} \ln L\end{aligned}\quad (65)$$

The disorder-averaged exponents $\tau_{av}(q)$ defined in Eq. 2 can be thus identified by the following expansion

$$\overline{Y_q(L)} \sim L^{-\tau_{av}(q)} = 1 - \tau_{av}(q) \ln L \quad (66)$$

This yields

$$\tau_{av}(q > \frac{1}{2}) = \frac{\Gamma\left(q - \frac{1}{2}\right)}{\Gamma(q-1)} \frac{c\sqrt{\pi} S_d V}{W} \overline{|u_{\vec{r}}|} \quad (67)$$

This expression coincides with Eq. 8 obtained previously via Levitov renormalization [9] for the PRBM model with the correspondence $b = V/W$ if one uses $c = \sqrt{\frac{2}{\pi}}$ (Eq. 13) and

$$\overline{|u|} = \int_{-\infty}^{+\infty} \frac{du}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} |u| = \sqrt{\frac{2}{\pi}} \quad (68)$$

with $d = 1$ and $S_1 = 2$ (to take into account the ring geometry).

VI. SUMMARY OF THE RESULTS AND DISCUSSION

In this section, we summarize and discuss the results obtained for the typical and the averaged multifractal spectra starting from the perturbation formula of Eq. 21.

A. Typical and averaged multifractal spectra $\tau_{typ}(q)$ and $\tau_{av}(q)$

In the region $q < 1/2$, we have found that the typical and the averaged multifractal spectra coincide (Eqs 51 and 64) and are given by the 'strong multifractality' limit of Eq. 6

$$\tau_{typ}(q < 1/2) = \tau_{av}(q < 1/2) = d(2q - 1) \quad (69)$$

In the region $q > 1/2$, we have found that the typical and the averaged multifractal spectra are different (Eqs 53 and 67)

$$\begin{aligned} \tau_{typ}(q > 1/2) &= S_d \frac{cV}{W} \overline{|u_{\vec{r}}|} \pi \left[q - \frac{1}{\sin(\frac{\pi}{2q})} \right] \\ \tau_{av}(q > 1/2) &= S_d \frac{cV}{W} \overline{|u_{\vec{r}}|} \sqrt{\pi} \frac{\Gamma(q - \frac{1}{2})}{\Gamma(q - 1)} \end{aligned} \quad (70)$$

where the results for $\tau_{av}(q > 1/2)$ coincides with the result of Eq. 8 obtained previously via Levitov renormalization [9].

Our conclusion is thus that the critical value q_c where the typical and averaged spectra separate (Eq. 4) is

$$q_c = \frac{1}{2} \quad (71)$$

in contrast with the other value $q_c \simeq 2.4$ predicted in [9], on the basis of the vanishing of the averaged singularity spectrum $f_{av}(\alpha)$ (see below Eq. 81). Moreover, we expect that the distribution $P_q(y_q)$ of the rescaled variable $y_q = Y_q(L)/Y_q^{typ}(L)$ will decay in the scaling regime with the power law of Eq. 3 of exponent $(1 + \beta_q)$ that will coincide with the exponent $(1 + \mu_q)$ describing the distribution of Σ_q (see Eq. 36)

$$\beta_q = \mu_q = \frac{1}{2q} = \frac{q_c}{q} \quad (72)$$

in agreement with Eq. 5.

B. Consequences for the typical and averaged singularity spectra $f_{typ}(\alpha)$ and $f_{av}(\alpha)$

Besides the multifractal exponents $\tau_{typ}(q)$ and $\tau_{av}(q)$, it is usual to introduce the typical and averaged singularity spectra $f_{typ}(\alpha)$ and $f_{av}(\alpha)$ which describe the numbers $\mathcal{N}_L^{typ,av}(\alpha)$ of points \vec{r} in a sample of size L^d , where the weight $|\psi_L(\vec{r})|^2$ scales as $L^{-\alpha}$ [8]

$$\mathcal{N}_L^{typ,av}(\alpha) \underset{L \rightarrow \infty}{\propto} L^{f_{typ,av}(\alpha)} \quad (73)$$

The I.P.R. Y_q can be then rewritten as integrals over α

$$Y_q^{typ,av}(L) \simeq \int d\alpha L^{f_{typ,av}(\alpha)} L^{-q\alpha} \underset{L \rightarrow \infty}{\simeq} L^{-\tau_{typ,av}(q)} \quad (74)$$

The exponents $\tau_{typ,av}(q)$ can be obtained via a saddle-point calculation in α to obtain the Legendre transform formula [6, 8]

$$-\tau_{typ,av}(q) = \max_{\alpha} [f_{typ,av}(\alpha) - q\alpha] \quad (75)$$

At leading order, the 'strong multifractality' limit of Eq. 6 (or Eq. 69 above) corresponds to the singularity spectra [9]

$$f_{typ}(\alpha) = f_{av}(\alpha) = \frac{\alpha}{2} \text{ for } 0 \leq \alpha \leq 2d \quad (76)$$

The typical exponent α_{typ} where $f_{typ}(\alpha_{typ}) = d$ thus corresponds to the maximal value $\alpha_{typ} = \alpha_{max} = 2d$. The singularity spectrum vanishes only at the other boundary $\alpha_{min} = 0$. From the point of view of the Legendre transform

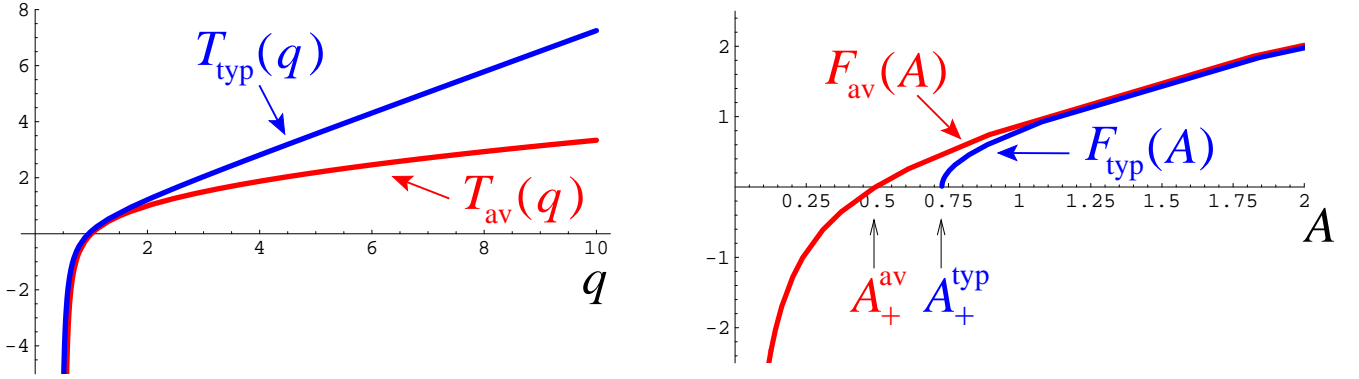


FIG. 1: (Color on line) (a) Comparison of the typical and averaged multifractal spectra $T_{typ}(q > 1/2)$ and $T_{av}(q > 1/2)$ of Eq. 77 : they are close near $q \rightarrow (1/2)^+$ (Eq. 78) but are very different at large q (Eq. 79). (b) Comparison of the corresponding typical and averaged singularity spectra $F_{typ}(A)$ and $F_{av}(A)$: $F_{typ}(A)$ exactly terminates at the point A_+^{typ} of Eq. 83 corresponding to $q = +\infty$, where it vanishes $F_{typ}(A_+^{typ}) = 0$, whereas $F_{av}(A)$ vanishes at another value A_+^{av} corresponding to q_+^{av} (see Eq. 81) and continues in the negative domain $F_{av}(A) < 0$.

of Eq. 75, this case is singular since the saddle $\alpha^*(q)$ is concentrated on the two values $\alpha^*(q < 1/2) = \alpha_{max} = 2d$ and $\alpha^*(q > 1/2) = \alpha_{min} = 0$.

Let us now take into account the small corrections of Eq. 70 in the domain $q > 1/2$, where we factorize the small prefactor $\epsilon = S_d \frac{c^V}{W} |\overline{u_r}|^{\frac{\pi}{2}}$ (this choice of constants in ϵ has been preferred to have the same normalization of $T_{av}(q)$ as in [9])

$$\begin{aligned} T_{typ}(q > 1/2) &\equiv \frac{\tau_{typ}(q > 1/2)}{\epsilon} = 2 \left[q - \frac{1}{\sin(\frac{\pi}{2q})} \right] \\ T_{av}(q > 1/2) &\equiv \frac{\tau_{av}(q > 1/2)}{\epsilon} = \frac{2\Gamma(q - \frac{1}{2})}{\sqrt{\pi}\Gamma(q - 1)} \end{aligned} \quad (77)$$

The two multifractal spectra $T_{typ}(q > 1/2)$ and $T_{av}(q > 1/2)$ are shown on Fig. 1 (a) for comparison : there are very close in the region $q \rightarrow 1/2$ where they present the same singularity

$$\begin{aligned} T_{typ}(q > 1/2) &\underset{q \rightarrow (1/2)^+}{\simeq} -\frac{1}{\pi(q - \frac{1}{2})} \\ T_{av}(q > 1/2) &\underset{q \rightarrow (1/2)^+}{\simeq} -\frac{1}{\pi(q - \frac{1}{2})} \end{aligned} \quad (78)$$

but are completely different at large q with the following asymptotic behaviors

$$\begin{aligned} T_{typ}(q > 1/2) &\underset{q \rightarrow +\infty}{\simeq} 2 \left(1 - \frac{2}{\pi} \right) q \\ T_{av}(q > 1/2) &\underset{q \rightarrow +\infty}{\simeq} \frac{2}{\sqrt{\pi}} q^{1/2} \end{aligned} \quad (79)$$

As discussed in [9], the disorder-averaged singularity spectrum $f_{av}(\alpha)$ takes the scaling form

$$f_{av}(\alpha) = \epsilon F_{av} \left(A = \frac{\alpha}{\epsilon} \right) \quad (80)$$

where $F_{av}(A)$ is the Legendre transform of $T_{av}(q)$: its properties have been described in [9]. In particular, it vanishes at A_+^{av} corresponding to q_+^{av} with the numerical values [9]

$$\begin{aligned} A_+^{av} &\simeq 0.51 \\ q_+^{av} &\simeq 2.405 \end{aligned} \quad (81)$$

Similarly, the typical singularity spectrum $f_{typ}(\alpha)$ takes the scaling form

$$f_{typ}(\alpha) = \epsilon F_{typ}\left(A = \frac{\alpha}{\epsilon}\right) \quad (82)$$

where $F_{typ}(A)$ is the Legendre transform of $T_{typ}(q)$. From the asymptotic linear behavior of $T_{typ}(q)$ (see Eq. 79), one obtains that $F_{typ}(A)$ exactly terminates at the point

$$A_+^{typ} = 2 \left(1 - \frac{2}{\pi}\right) \simeq 0.727 \quad (83)$$

corresponding to $q = +\infty$, where it vanishes $F_{typ}(A_+^{typ}) = 0$. The two singularity spectra $F_{av}(A)$ and $F_{typ}(A)$ are shown in Fig. 1 (b) for comparison.

The fact that $F_{typ}(A)$ exists only in the region where it remains positive $F_{typ}(A) \geq 0$ is a standard property of typical spectrum [8]. What is surprising however is that the typical and averaged singularity spectra differ even in the region where there are positive, whereas the standard picture is that $F_{typ}(A) = F_{av}(A)\theta(F_{av}(A) > 0)$ [8]. Equivalently, in this standard picture [8], the typical spectrum is expected to be exactly linear $T_{typ}(q) = A_+q$ for $q > q_+$ meaning that in the Legendre calculation of Eq. 75, the saddle value remains frozen at $A_{typ}(q \geq q_+) = A_+$. Here we have found instead that the typical spectrum $T_{typ}(q)$ is not exactly linear in q in the region where it is different from $T_{av}(q)$, and that the saddle value $A_{typ}(q)$ of the Legendre transform of Eq. 75 reaches the termination point A_+^{typ} only in the asymptotic regime $q \rightarrow +\infty$.

C. Discussion on the method

The perturbative calculation of the multifractal spectrum in the strong multifractality regime can only be very singular since one starts from a complete localized basis to construct multifractal critical eigenvectors via perturbation. To face this difficulty, two strategies have been proposed :

(i) The powerful Levitov renormalization method [9, 30, 31] performs iterative changes of bases to take into account the resonances that occur at various scales. This approach has been reformulated as some type of 'virial expansion' in Refs [32–35].

(ii) In the present paper, we have proposed instead to use the standard perturbation theory of quantum mechanics. It is thus simpler than (i), since we work in the initial completely localized basis. However, since the random perturbation terms are singular, the essential point in this approach is that the I.P.R. should be computed with Eq. 21, where the perturbation terms of the eigenvectors appear both in the numerator and in the denominator : this ratio is then regular, because any potential divergence appearing in the numerator is compensated by the corresponding divergence in the denominator. In the present paper, we have described in detail how the first-order expression of the perturbed eigenvector allows to obtain the leading order of the multifractal spectrum in various regions of q . A natural question is how this approach can be pursued at higher orders. We stress that one should not use the standard Rayleigh-Schrödinger expressions for the normalized perturbed eigenvector (since these expressions are in fact based on the perturbative expansion of the normalization, which is singular here as explained above). We believe that the correct formulation of our approach at higher orders involve the perturbative expansion of the eigenvector

$$|\phi_{\vec{r}}^{tot} \rangle = |\phi_{\vec{r}}^{(0)} \rangle + \sum_{n=1}^{+\infty} |\phi_{\vec{r}}^{(n)} \rangle = |\vec{r} \rangle + \sum_{n=1}^{+\infty} |\phi_{\vec{r}}^{(n)} \rangle \quad (84)$$

in the so-called intermediate normalization defined by

$$\langle \phi_{\vec{r}}^{(0)} | \phi_{\vec{r}}^{tot} \rangle = \langle \vec{r} | \phi_{\vec{r}}^{tot} \rangle = 1 \quad (85)$$

so that all corrections are orthogonal to the zeroth order term $|\phi_{\vec{r}}^{(0)} \rangle = |\vec{r} \rangle$

$$\langle \phi_{\vec{r}}^{(0)} | \phi_{\vec{r}}^{(n)} \rangle = \langle \vec{r} | \phi_{\vec{r}}^{(n)} \rangle = 0 \quad \text{for } n \geq 1 \quad (86)$$

The corresponding I.P.R. of Eq. 1 should be then obtained as

$$Y_q = \frac{\sum_{\vec{r}'} |\phi_{\vec{r}'}^{tot}(\vec{r}')|^{2q}}{\left[\sum_{\vec{r}'} |\phi_{\vec{r}'}^{tot}(\vec{r}')|^2 \right]^q} = \frac{1 + \sum_{\vec{r}' \neq \vec{r}} |\phi_{\vec{r}}^{tot}(\vec{r}')|^{2q}}{\left(1 + \sum_{\vec{r}' \neq \vec{r}} |\phi_{\vec{r}}^{tot}(\vec{r}')|^2 \right)^q} \quad (87)$$

VII. CONCLUSIONS AND PERSPECTIVES

In summary, we have show that that the strong multifractality regime of Anderson tight-binding models in dimension d with critical long-ranged hoppings can be studied via the standard perturbation theory for eigenvectors in quantum mechanics. The Inverse Participation Ratios $Y_q(L)$, which are the order parameters of Anderson transitions, then involve weighted Lévy sums of broadly distributed variables, as a consequence of the presence of on-site random energies in the denominators of the perturbation theory. We have computed at leading order the typical and disorder-averaged multifractal spectra $\tau_{typ}(q)$ and $\tau_{av}(q)$ as a function of q . For $q < 1/2$, we have found the non-vanishing limiting spectrum $\tau_{typ}(q) = \tau_{av}(q) = d(2q - 1)$ as $V \rightarrow 0^+$, that had been obtained previously in [9] only indirectly via the symmetry relation of Eq. 9. For $q > 1/2$, we have obtained the same result for disorder-averaged spectrum $\tau_{av}(q)$ at order $O(V)$ as obtained previously via the Levitov renormalization method [9]. This agreement between these two completely different approaches is a good indication in favor of the exactness of this result. But in addition, our present approach allows to compute explicitly the typical spectrum (that has not been computed via Levitov renormalization) : it is also of order $O(V)$ but has a different q -dependence $\tau_{typ}(q) \neq \tau_{av}(q)$ for $q > q_c = 1/2$, and is not exactly linear in this regime, in contrast with the standard picture [8]. As a consequence, we have found that the corresponding singularity spectra $f_{typ}(\alpha)$ and $f_{av}(\alpha)$ differ even in the positive region $f > 0$, in contrast with the standard picture where they coincide in the positive region $f_{typ}(\alpha) = f_{av}(\alpha)\theta(f_{av}(\alpha) > 0)$ [8], and that the saddle value $A_{typ}(q)$ of the Legendre transform reaches the termination point A_+^{typ} where $f_{typ}(A_+^{typ}) = 0$ only in the limit $q \rightarrow +\infty$.

In conclusion, the present work based on a pedestrian perturbative explicit approach thus questioned important statements of the standard picture of multifractality at Anderson transitions. We hope that it will stimulate further studies to better understand the differences between typical and averaged multifractal spectra.

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- [1] P.W. Anderson, Phys. Rev. 109, 1492 (1958).
 - [2] D.J. Thouless, Phys. Rep. 13, 93 (1974) ; D.J. Thouless, in “Ill Condensed Matter” (Les Houches 1978), Eds R. Balian *et al.* North-Holland, Amsterdam (1979).
 - [3] B. Souillard, in “Chance and Matter” (Les Houches 1986), Eds J. Souletie *et al.* North-Holland, Amsterdam (1987).
 - [4] I.M. Lifshitz, S.A. Gredeskul and L.A. Pastur, “Introduction to the theory of disordered systems” (Wiley, NY, 1988).
 - [5] B. Kramer and A. MacKinnon, Rep. Prog. Phys. 56, 1469 (1993).
 - [6] M. Janssen, Phys. Rep. 295, 1 (1998).
 - [7] P. Markos, Acta Physica Slovaca 56, 561 (2006).
 - [8] F. Evers and A.D. Mirlin, Rev. Mod. Phys. 80, 1355 (2008).
 - [9] F. Evers and A. D. Mirlin Phys. Rev. Lett. 84, 3690 (2000); A.D. Mirlin and F. Evers, Phys. Rev. B 62, 7920 (2000).
 - [10] C. Monthus and T. Garel, arXiv:1004.1537
 - [11] A. Mildenberger, F. Evers and A.D. Mirlin, Phys. Rev. B 66, 033109 (2002).
 - [12] F. Wegner, Nucl. Phys. B 280, 210 (1987).
 - [13] A. D. Mirlin, Y.V. Fyodorov, A. Mildenberger and F. Evers, Phys. Rev. Lett 97, 046803 (2006).
 - [14] A. Mildenberger and F. Evers, Phys. Rev. B 75, 041303(R) (2007).
 - [15] F. Evers, A. Mildenberger, A. D. Mirlin, Physica Status Solidi B 245, 284 (2008); F. Evers, A. Mildenberger, and A. D. Mirlin, Phys. Rev. Lett. 101, 116803 (2008).
 - [16] L.J. Vasquez, A. Rodriguez and R.A. Romer, Phys. Rev. B 78, 195106 (2008); A. Rodriguez, L.J. Vasquez and R.A. Romer, Phys. Rev. B 78, 195107 (2008); A. Rodriguez, L.J. Vasquez and R.A. Romer, Phys. Rev. Lett. 102, 106406 (2009).
 - [17] C. Monthus, B. Berche and C. Chatelain, J. Stat. Mech. P12002 (2009).
 - [18] A.D. Mirlin, Y.V. Fyodorov, F.M. Dittes, J. Quezada and T. Seligman, Phys. Rev. E 54, 3221 (1996).
 - [19] I. Varga and D. Braun, Phys. Rev. B 61, R11859 (2000).
 - [20] V.E. Kravtsov *et al.*, J. Phys. A 39, 2021 (2006).
 - [21] A.M. Garcia-Garcia, Phys. Rev. E 73, 026213 (2006).
 - [22] E. Cuevas, V. Gasparian and M. Ortuno, Phys. Rev. Lett. 87, 056601 (2001).
 - [23] E. Cuevas *et al.*, Phys. Rev. Lett. 88, 016401 (2001).
 - [24] I. Varga, Phys. Rev. B 66, 094201 (2002).
 - [25] E. Cuevas, Phys. Rev. B 68, 024206 (2003).
 - [26] A. Mildenberger *et al.*, Phys. Rev. B 75, 094204 (2007).

- [27] J.A. Mendez-Bermudez and T. Kottos, Phys. Rev. B 72, 064108 (2005).
- [28] J.A. Mendez-Bermudez and I. Varga, Phys. Rev. B 74, 125114 (2006).
- [29] C. Monthus and T. Garel, Phys. Rev. B 79, 205120 (2009); C. Monthus and T. Garel, J. Stat. Mech. P07033 (2009).
- [30] L.S. Levitov, Europhys. Lett. 9, 83 (1989); L.S. Levitov, Phys. Rev. Lett. 64, 547 (1990); B.L. Altshuler and L.S. Levitov, Phys. Rep. 288, 487 (1997); L.S. Levitov, Ann. Phys. (Leipzig) 8, 5, 507 (1999).
- [31] Y.V. Fyodorov, A. Ossipov and A. Rodriguez, J. Stat. Mech. L12001 (2009).
- [32] O. Yevtushenko and V. E. Kratsov, J. Phys. A 36, 8265 (2003).
- [33] O. Yevtushenko and A. Ossipov, J. Phys. A 40, 4691 (2007).
- [34] S. Kronmüller, O. M. Yevtushenko and E. Cuevas, J. Phys. A 43, 075001 (2010).
- [35] V. E. Kratsov, A. Ossipov, O. M. Yevtushenko and E. Cuevas, arXiv:1008.2694
- [36] J.P. Bouchaud and A. Georges, Phys. Rep. 195, 127 (1990).
- [37] B. Derrida, Physica D 107, 186 (1997).
- [38] B. Derrida and H. Flyvbjerg, J. Phys. A Math. Gen. 20, 5273 (1987).
- [39] C. Monthus and T. Garel, Phys. Rev. E 75, 051119 (2007).